Real-time Head Pose Tracking with Online Face Template Reconstruction - *Supplementary Material*

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1. Calculation of R and t

In this section, we derive the closed form solution of **R** and **t** for minimizing the following objective function

$$c_1(\mathbf{R},\mathbf{t}) = \sum_{i=1}^{L} w_i[(\mathbf{R}\mathbf{n}_i)\cdot(\mathbf{R}(\boldsymbol{\mu}_i+\mathbf{P}_i\boldsymbol{\alpha})+\mathbf{t}-\mathbf{d}_i)]^2 \tag{1}$$

Firstly, eq. (1) can be reformulated as

$$\Sigma_{i=1}^{L} w_i(\mathbf{f}_i+\mathbf{R}^\top\mathbf{t}-\mathbf{R}^\top\mathbf{d}_i)^\top\mathbf{n}_i\mathbf{n}_i^\top(\mathbf{f}_i+\mathbf{R}^\top\mathbf{t}-\mathbf{R}^\top\mathbf{d}_i) \tag{2}$$

with $\mathbf{f}_i$ given by

$$\mathbf{f}_i = \boldsymbol{\mu}_i + \mathbf{P}_i\boldsymbol{\alpha} \tag{3}$$

Let $\mathbf{W}_i = w_i\mathbf{n}_i\mathbf{n}_i^\top$, $\widetilde{\mathbf{R}} = \mathbf{R}^\top$, and $\widetilde{\mathbf{t}} = \widetilde{\mathbf{R}}\mathbf{t}$, we can write eq. (2) as

$$\Sigma_{i=1}^{L}(\mathbf{f}_i+\widetilde{\mathbf{t}}-\widetilde{\mathbf{R}}\mathbf{d}_i)^\top\mathbf{W}_i(\mathbf{f}_i+\widetilde{\mathbf{t}}-\widetilde{\mathbf{R}}\mathbf{d}_i) \tag{4}$$

We decompose the rotation matrix $\widetilde{\mathbf{R}}$ into an initial rotation matrix $\widetilde{\mathbf{R}}_0$ (the rotation matrix derived in the last iteration) and an incremental rotation matrix $\Delta\widetilde{\mathbf{R}}$, i.e., $\widetilde{\mathbf{R}} = \Delta\widetilde{\mathbf{R}}\widetilde{\mathbf{R}}_0$. Assume that the rotation angles (ω_1, ω_2, ω_3) of $\Delta\widetilde{\mathbf{R}}$ are small, we have

$$\Delta\widetilde{\mathbf{R}} = \begin{bmatrix} 1 & -\omega_3 & \omega_2 \\ -\omega_3 & 1 & -\omega_1 \\ \omega_2 & -\omega_1 & 1 \end{bmatrix} \tag{5}$$

Let $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^\top$, $\mathbf{q}_i = \widetilde{\mathbf{R}}_0\mathbf{d}_i = [q_{i_1}, q_{i_2}, q_{i_3}]^\top$, and $[\mathbf{q}_i]_\times$ denotes the skew-symmetric matrix of $\mathbf{q}_i$, i.e.,

$$[\mathbf{q}_i]_\times = \begin{bmatrix} 0 & -q_{i_3} & q_{i_2} \\ -q_{i_3} & 0 & -q_{i_1} \\ q_{i_2} & -q_{i_1} & 0 \end{bmatrix} \tag{6}$$

we can write eq. (4) as

$$\Sigma_{i=1}^{L}(\mathbf{f}_i - \mathbf{q}_i + [I_3, [\mathbf{q}_i]_\times] \begin{bmatrix} \widetilde{\mathbf{t}} \\ \boldsymbol{\omega} \end{bmatrix})^\top \mathbf{W}_i(\mathbf{f}_i - \mathbf{q}_i + [I_3, [\mathbf{q}_i]_\times] \begin{bmatrix} \widetilde{\mathbf{t}} \\ \boldsymbol{\omega} \end{bmatrix}) \tag{7}$$

which is a quadratic function with respect to the unknowns $[\widetilde{\mathbf{t}}, \boldsymbol{\omega}]^\top$. Therefore, to minimize eq. (7), the unknowns can be calculated analytically as

$$\begin{bmatrix} \widetilde{\mathbf{t}} \\ \boldsymbol{\omega} \end{bmatrix} = -(\Sigma_{i=1}^{L}\mathbf{A}_i^\top \mathbf{W}_i \mathbf{A}_i)^{-1}(\Sigma_{i=1}^{L}\mathbf{A}_i^\top \mathbf{W}_i^\top \mathbf{b}_i) \tag{8}$$

where $\mathbf{A}_i = [I_3, [\mathbf{q}_i]_\times]$, and $\mathbf{b}_i = \mathbf{f}_i - \mathbf{q}_i$. Given $\widetilde{\mathbf{t}}$ and $\boldsymbol{\omega}$, we can calculate $\Delta\widetilde{\mathbf{R}}$, $\widetilde{\mathbf{R}}$, and then $\mathbf{R}$ and $\mathbf{t}$. Notice that singular value decomposition is needed to convert the $\Delta\widetilde{\mathbf{R}}$ given by eq. (5) into a real rotation matrix.

2. Calculation of $\boldsymbol{\alpha}$

In this section, we derive the closed form solution of $\boldsymbol{\alpha}$ which minimizes minimizing the following objective function

$$c_2(\boldsymbol{\alpha}) = \sum_{i=1}^{L} w_i[(\mathbf{R}\mathbf{n}_i) \cdot (\mathbf{R}(\boldsymbol{\mu}_i + \mathbf{P}_i\boldsymbol{\alpha}) + \mathbf{t} - \mathbf{d}_i)]^2 + \lambda \sum_{j=1}^{K} \frac{\alpha_j^2}{\sigma_j^2} \tag{9}$$

Firstly, eq. (9) can be reformulated as

$$\Sigma_{i=1}^{L} w_i[\mathbf{n}_i^\top(\mathbf{P}_i\boldsymbol{\alpha} + \mathbf{R}^\top(\mathbf{t} - \mathbf{d}_i) + \boldsymbol{\mu}_i)]^2 + \lambda\Sigma_{j=1}^{K}\frac{\alpha_j^2}{\sigma_j^2} \tag{10}$$

Let

$$\mathbf{g}_i = \mathbf{R}^\top(\mathbf{t} - \mathbf{d}_i) + \boldsymbol{\mu}_i \tag{11}$$

and

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{\sigma_1^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sigma_K^2} \end{bmatrix} \tag{12}$$

we can write eq. (10) as

$$\Sigma_{i=1}^{L} w_i [\mathbf{n}_i^\top (\mathbf{P}_i \boldsymbol{\alpha} + \mathbf{g}_i)]^2 + \lambda \boldsymbol{\alpha}^\top \mathbf{Q} \boldsymbol{\alpha} \tag{13}$$

which is equivalent to

$$\Sigma_{i=1}^{L} w_i (\mathbf{P}_i \boldsymbol{\alpha} + \mathbf{g}_i)^\top \mathbf{n}_i \mathbf{n}_i^\top (\mathbf{P}_i \boldsymbol{\alpha} + \mathbf{g}_i) + \lambda \boldsymbol{\alpha}^\top \mathbf{Q} \boldsymbol{\alpha} \tag{14}$$

Now Let

$$\mathbf{W}_i = w_i \mathbf{n}_i \mathbf{n}_i{}^\top \tag{15}$$

we can write eq. (14) as

$$\Sigma_{i=1}^{L} (\mathbf{P}_i \boldsymbol{\alpha} + \mathbf{g}_i)^\top \mathbf{W}_i (\mathbf{P}_i \boldsymbol{\alpha} + \mathbf{g}_i) + \lambda \boldsymbol{\alpha}^\top \mathbf{Q} \boldsymbol{\alpha} \tag{16}$$

which after decomposition is

$$\boldsymbol{\alpha}^\top (\Sigma_{i=1}^{L} \mathbf{P}_i^\top \mathbf{W}_i \mathbf{P}_i + \lambda \mathbf{Q}) \boldsymbol{\alpha} + 2 (\Sigma_{i=1}^{L} \mathbf{g}_i^\top \mathbf{W}_i \mathbf{P}_i) \boldsymbol{\alpha} + \Sigma_{i=1}^{L} \mathbf{g}_i^\top \mathbf{W}_i \mathbf{g}_i \tag{17}$$

It can be seen that eq. (17) is a quadratic function with respect to the unknown $\boldsymbol{\alpha}$. Therefore, its minimum solution can be given by

$$\boldsymbol{\alpha} = -(\sum_{i=1}^{M} \mathbf{P}_i{}^\top \mathbf{W}_i \mathbf{P}_i + \lambda \mathbf{Q})^{-1} (\sum_{i=1}^{M} \mathbf{P}_i{}^\top \mathbf{W}_i{}^\top \mathbf{g}_i) \tag{18}$$