QUIZ NUMBER 4
November 19, 2008

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This is a closed-book examination (duration: 40 minutes).
Answer the questions in the spaces provided on the question sheets.
SIMPLE NUMERICAL RESULTS ARE NOT SUFFICIENT: always justify your answers.

Question 1 [7 points]: Magnetostatics
A current density \( J \) generates a static magnetic flux density \( B \) in a space of permeability \( \mu_0 \).

(a) [2 points] What differential equations should the field \( B \) satisfy?

Solution:
\[
\text{div } B = 0 \quad \text{and} \quad \nabla \times B = \mu_0 J
\]

(b) [3 points] What is the magnetic potential vector? Is it uniquely defined? What is the usual choice for this vector?

Solution: The magnetic potential vector \( A \) is such that
\[
B = \nabla \times A
\]
This vector is not unique (it is defined up to a gradient \( \nabla \varphi \), because \( \nabla \times \nabla \varphi = 0 \)). The usual choice of \( A \) is the one for which \( \text{div } A = 0 \).

(c) [2 points] What is the integral solution for the magnetic potential vector in function of \( J \)?

Solution:
\[
A(r) = \frac{\mu_0}{4\pi} \int \frac{J(r')}{|r - r'|} \, dx'\, dy'\, dz'
\]

Question 2 [12 points]: Maxwell’s equations
We consider the propagation of electromagnetic waves \( E(x, y, z, t) \) and \( H(x, y, z, t) \) in the vacuum (permittivity \( \varepsilon_0 \) and permeability \( \mu_0 \)).

(a) [3 points] What are the differential equations that characterize the electromagnetic field?

Solution:
\[
\begin{align*}
\text{div } E &= 0 \\
\nabla \times E &= -\mu_0 \frac{\partial H}{\partial t} \\
\text{div } H &= 0 \\
\nabla \times H &= \varepsilon_0 \frac{\partial E}{\partial t}
\end{align*}
\]
(b) [3 points] By using the identity $\nabla \times (\nabla \times \mathbf{u}) = -\nabla^2 \mathbf{u} + \nabla (\text{div} \, \mathbf{u})$ show that $\mathbf{E}$ and $\mathbf{H}$ satisfy the equations

$$\nabla^2 \mathbf{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{H} - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

**Solution:** By taking the curl of Faraday’s equation

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \nabla \times \frac{\partial \mathbf{H}}{\partial t}$$

$$= -\mu_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{H})$$

$$= -\mu_0 \frac{\partial}{\partial t} \left( \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

by Ampère’s circuital law

On the other hand

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} + \nabla (\text{div} \, \mathbf{E})$$

hence $-\nabla^2 \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$. Same proof for $\mathbf{H}$.

(c) We assume that $\mathbf{E}(x, y, z, t)$ and $\mathbf{H}(x, y, z, t)$ can be expressed as

$$\mathbf{E}(x, y, z, t) = \mathbf{E}_0 (z - ct)$$

and

$$\mathbf{H}(x, y, z, t) = \mathbf{H}_0 (z - ct)$$

where $c$ is some constant. We also assume that there is no (electro- or magneto-) static field.

i. [1 point] What is the value of $c$ in function of $\varepsilon_0$ and $\mu_0$?

**Solution:** $c$ is the propagation velocity of the EM field because

$$\nabla^2 \mathbf{E} = \mathbf{E}_0''(z - ct) = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Thus, $c = 1/\sqrt{\varepsilon_0 \mu_0}$. 

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**ELE 3310 BASIC ELECTROMAGNETIC THEORY**  
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ii. [3 points] Show that

\[ \nabla \times E = -\frac{1}{c} \frac{\partial}{\partial t} (a_z \times E) \]

Solution:

\[
\nabla \times E = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
E_{0x}(z - ct) & E_{0y}(z - ct) & E_{0z}(z - ct) \\
\end{vmatrix}
= \frac{\partial}{\partial y} E_{0z}(z - ct) - \frac{\partial}{\partial z} E_{0y}(z - ct)
\]

\[ = -\frac{1}{c} \frac{\partial}{\partial t} (a_z \times E) \]

hence the result.

iii. [2 points] What is the expression of the magnetic field intensity \( H \) in function of \( E \)?

Solution:

\[
\nabla \times E = -\mu_0 \frac{\partial H}{\partial t}
\]

\[ = -\frac{1}{c} \frac{\partial}{\partial t} (a_z \times E) \]

Hence \( \mu_0 H = \frac{1}{c} a_z \times E + \text{static field. However, by hypothesis, there is no static field. This implies that} \)

\[ H = \frac{1}{\mu_0 c} a_z \times E \]

**Question 3 [6 points]: Phasors and plane waves**

We consider phasors at the radial frequency \( \omega \)

(a) [2 points] Give the expression of the electric field \( E(x, y, z, t) \) in function of its phasor \( E_0(x, y, z) \)

Solution:

\[ E(x, y, z, t) = \Re \{ E_0(x, y, z)e^{j\omega t} \} \]

(b) [2 points] If we assume that the phasor \( E_0(x, y, z) = a_x e^{-jkz} \), give the expression of the magnetic flux density phasor \( B_0(x, y, z) \).

Solution: Using Faraday’s equation for phasors,

\[ \nabla \times E(x, y, z) = -jk a_z \times a_x e^{j\omega t - jkz} = -j\omega B_0 \]

Hence

\[ B_0 = \frac{k}{\omega} a_y e^{-jkz} \]
(c) [2 points] In a dielectric medium (permittivity \( \varepsilon = 2 \times 10^{-10} \text{ F/m} \), permeability \( \mu = 2 \times 10^{-6} \text{ H/m} \)), if we consider an EM wave at the frequency 100 GHz, give the value of the wave velocity, the wave impedance and the wavelength.

Solution:

<table>
<thead>
<tr>
<th>wave velocity</th>
<th>wave impedance</th>
<th>wavelength</th>
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<tbody>
<tr>
<td>( c = \frac{1}{\sqrt{\varepsilon \mu}} )</td>
<td>( \eta = \sqrt{\frac{\mu}{\varepsilon}} )</td>
<td>( \lambda = \frac{c}{f} )</td>
</tr>
<tr>
<td>( = 5 \times 10^7 \text{ m/s} )</td>
<td>( = 100 \Omega )</td>
<td>( = 5 \times 10^{-4} \text{ m} )</td>
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