2.4 Electric Potential

Electric Potential

Since \( \nabla \times \nabla \Phi \equiv 0 \) and \( \nabla \times \vec{E} \equiv 0 \)

We can define a scalar electric potential \( V \) such that

\[
\vec{E} = -\nabla V
\]

Integrating the \( \vec{E} \) field from point \( P_1 \) to \( P_2 \) in an electric field

\[
- \int_{P_1}^{P_2} \vec{E} \cdot d\vec{l} = \int_{P_1}^{P_2} \nabla V \cdot d\vec{l} = \int_{P_1}^{P_2} \frac{dV}{dl} \hat{a}_l \cdot \hat{a}_l dl = \int_{P_1}^{P_2} dV = V_2 - V_1
\]

When the reference zero-potential point is not at infinity, it should be specifically stated.
2.4 Electric Potential (cont.)

Potential of charge

The electric field intensity $\mathbf{E}$ of a point charge $q$ at a distance $R$ is

$$\mathbf{E} = \hat{a}_r E_R = \hat{a}_r \frac{q}{4\pi\varepsilon_0 R^2} \quad (V/m)$$

The potential of a point at a distance $R$ from a point charge $q$ referred to that at infinity can be obtained readily

which gives

$$V = -\int_{\infty}^{R} \left(\hat{a}_r \frac{q}{4\pi\varepsilon_0 R^2}\right) \cdot (\hat{a}_r dR)$$

or

$$V = \frac{q}{4\pi\varepsilon_0 R}$$
2.4 Electric Potential (cont.)

For multiple point charge case

\[ V = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{q_k}{|R - R'_k|} \]

For the electric potential due to a continuous distribution of charge, we have

\[ V = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho}{R} \, dv' \]

3D case

\[ V = \frac{1}{4\pi\varepsilon_0} \int_{S'} \frac{\rho_s}{R} \, ds' \]

2D case

\[ V = \frac{1}{4\pi\varepsilon_0} \int_{L'} \frac{\rho_l}{R} \, dl' \]

1D case
EXAMPLE 3–9  Obtain a formula for the electric field intensity on the axis of a circular disk of radius \( b \) that carries a uniform surface charge density \( \rho_s \).

Solution  Although the disk has circular symmetry, we cannot visualize a surface around it over which the normal component of \( \mathbf{E} \) has a constant magnitude; hence Gauss’s law is not useful for the solution of this problem. We use Eq. (3–62). Working with cylindrical coordinates indicated in Fig. 3–16, we have

\[
ds' = r' \, dr' \, d\phi'
\]

and

\[
R = \sqrt{z^2 + r'^2}.
\]

The electric potential at the point \( P(0, 0, z) \) referring to the point at infinity is

\[
V = \frac{\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^b \frac{r'}{(z^2 + r'^2)^{1/2}} \, dr' \, d\phi'
\]

\[
= \frac{\rho_s}{2\varepsilon_0} \left[ (z^2 + b^2)^{1/2} - |z| \right].
\]
Therefore,

\[
E = -\nabla V = -a_z \frac{\partial V}{\partial z}
\]

\[
= \begin{cases} 
  a_z \frac{\rho_s}{2\epsilon_0} \left[1 - z(z^2 + b^2)^{-1/2}\right], & z > 0 \\
  -a_z \frac{\rho_s}{2\epsilon_0} \left[1 + z(z^2 + b^2)^{-1/2}\right], & z < 0.
\end{cases}
\]

(3–65a)  

(3–65b)
2.4 Electric Potential (cont.)

The determination of \( \mathbf{E} \) field at an off-axis point would be a much more difficult problem. Do you know why?

For very large \( z \), it is convenient to expand the second term in Eqs. (3–65a) and (3–65b) into a binomial series and neglect the second and all higher powers of the ratio \( \frac{b^2}{z^2} \). We have

\[
z(z^2 + b^2)^{-1/2} = \left(1 + \frac{b^2}{z^2}\right)^{-1/2} \approx 1 - \frac{b^2}{2z^2}.
\]

Substituting this into Eqs. (3–65a) and (3–65b), we obtain

\[
\mathbf{E} = az \frac{\pi b^2 \rho_s}{4\pi \varepsilon_0 z^2}
\]

\[
= \begin{cases} 
  \frac{a_z Q}{4\pi \varepsilon_0 z^2}, & z > 0 \\
  -\frac{a_z Q}{4\pi \varepsilon_0 z^2}, & z < 0,
\end{cases}
\]

where \( Q \) is the total charge on the disk. Hence, when the point of observation is very far away from the charged disk, the \( \mathbf{E} \) field approximately follows the inverse square law as if the total charge were concentrated at a point.
Assume for the present that some positive (or negative) charges are introduced in the interior of a conductor. An electric field will be set up in the conductor, the field exerting a force on the charges and making them move away from one another. This movement will continue until all the charges reach the conductor surface and redistribute themselves in such a way that both the charge and the field inside vanish.

<table>
<thead>
<tr>
<th>Inside a Conductor (Under Static Conditions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 0$</td>
</tr>
<tr>
<td>$E = 0$</td>
</tr>
</tbody>
</table>

The charge distribution on the surface of a conductor depends on the shape of the surface. Obviously, tangential component of the electric field must be zero.

E field on a conductor surface is everywhere normal to the surface.
The presence of an external electric field causes a force to be exerted on each charged particle and results in small displacements of positive and negative charges in opposite directions. These displacements polarize a dielectric material and create electric dipoles, which equivalent to an volume charge density $\rho_p$.

The polarized charge density is related to the polarization vector $\mathbf{P}$ by

$$\rho_p = -\nabla \cdot \mathbf{P}$$

The “-” sign means that the field generated by the electric dipoles is in the opposite direction of external electric field $\mathbf{E}$.
The divergence postulation must be modified to include the effect of $\rho_p$

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} (\rho + \rho_p)$$

or

$$\nabla \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho$$

We now define a new fundamental field quantity, the electric flux density, or electric displacement, $D$, such that

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \quad (C / m^2)$$

or

$$\nabla \cdot \vec{D} = \rho \quad (C / m^3)$$
When the dielectric properties of the medium are linear and isotropic, the polarization is directly proportional to the electric field intensity, and the proportionality constant is independent of the direction of the field, We have

\[ \vec{P} = \varepsilon_0 \chi_e \vec{E} \]

where \( \chi_e \) is a dimensionless quantity called electric susceptibility. It is obvious that

\[ \vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \ (C/m^2) \]

### Table: Relative Permittivity, \( \varepsilon_r \)

<table>
<thead>
<tr>
<th>Material</th>
<th>Relative Permittivity, ( \varepsilon_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>1.0</td>
</tr>
<tr>
<td>Bakelite</td>
<td>5.0</td>
</tr>
<tr>
<td>Glass</td>
<td>4–10</td>
</tr>
<tr>
<td>Mica</td>
<td>6.0</td>
</tr>
<tr>
<td>Oil</td>
<td>2.3</td>
</tr>
<tr>
<td>Paper</td>
<td>2–4</td>
</tr>
<tr>
<td>Parafin wax</td>
<td>2.2</td>
</tr>
<tr>
<td>Plexiglass</td>
<td>3.4</td>
</tr>
<tr>
<td>Polyethylene</td>
<td>2.3</td>
</tr>
<tr>
<td>Polystyrene</td>
<td>2.6</td>
</tr>
<tr>
<td>Porcelain</td>
<td>5.7</td>
</tr>
<tr>
<td>Rubber</td>
<td>2.3–4</td>
</tr>
<tr>
<td>Soil (dry)</td>
<td>3–4</td>
</tr>
<tr>
<td>Teflon</td>
<td>2.1</td>
</tr>
<tr>
<td>Water (distilled)</td>
<td>80</td>
</tr>
<tr>
<td>Seawater</td>
<td>72</td>
</tr>
</tbody>
</table>