7.6 Normal incidence at a conducting Plane

Consider the situation in the figure where the incident wave travels in the +z-direction, and the boundary surface is the plane $z = 0$. The incident electric and magnetic field intensity phasors are

$$
E_i(z) = a_x E_{i0} e^{-j\beta_1 z}, \quad H_i(z) = a_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}
$$

where $E_{i0}$ is the magnitude of $E_i$ at $z = 0$, and $\beta_1$ and $\eta_1$ are the phase constant and the intrinsic impedance, respectively, of medium 1.

Poynting vector $P_i(z) = E_i(z) \times H_i(z)$ is in the $a_z$ direction
7.6 Normal incidence at a conducting Plane

Assume that the reflected electric field intensity can be written as

\[ E_r(z) = a_x E_{r0} e^{+j\beta_1 z} \]

Then the total electric field in medium 1 is

\[ E_1(z) = E_i(z) + E_r(z) = a_x (E_{i0} e^{-j\beta_1 z} + E_{r0} e^{+j\beta_1 z}) \]

The boundary condition of tangential \( E \) field at conducting surface is zero leads to

\[ E_1(0) = a_x (E_{i0} + E_{r0}) = E_2(0) = 0 \]

which yields \( E_{r0} = -E_{i0} \) or

\[ E_1(z) = a_x E_{i0} \left( e^{-j\beta_1 z} - e^{+j\beta_1 z} \right) = -a_x j2E_{i0} \sin \beta_1 z \]
7.6 Normal incidence at a conducting Plane

\[
E_1(z) = a_x E_{i0} \left( e^{-j \beta_1 z} - e^{+j \beta_1 z} \right) = -a_x j 2E_{i0} \sin \beta_1 z
\]

The magnetic field \(H_r\) of the reflected wave is related to \(E_r\) by

\[
H_r(z) = \frac{1}{\eta_1} a_y \times E_r(z) = \frac{1}{\eta_1} (-a_z) \times E_r(z)
\]

\[
= -a_y \frac{1}{\eta_1} E_{r0} e^{+j \beta_1 z} = a_y \frac{E_{i0}}{\eta_1} e^{+j \beta_1 z}
\]

The total magnetic field intensity in medium 1

\[
H_1(z) = H_i(z) + H_r(z) = a_y 2 \frac{E_{i0}}{\eta_1} \cos \beta_1 z
\]

It is clear that no average power is associated with the total electromagnetic wave in medium 1, since \(E_1(z)\) and \(H_1(z)\) are in phase quadrature.

The space-time behavior of the total field in medium 1 can be examined by

\[
E_1(z,t) = \mathcal{R}e \left[ E_1(z) e^{j \omega t} \right] = a_x 2 E_{i0} \sin \beta_1 z \sin \omega t
\]

\[
H_1(z,t) = \mathcal{R}e \left[ H_1(z) e^{j \omega t} \right] = a_y 2 \frac{E_{i0}}{\eta_1} \cos \beta_1 z \cos \omega t
\]

The total wave in medium 1 is not a traveling wave. It is a standing wave, resulting from the superposition of two waves traveling in opposite directions.
7.7 Oblique incidence at a conducting Plane

When a uniform plane wave is incident on a plane conducting surface obliquely. In the case of perpendicular polarization, $E_i$ is perpendicular to the plane of incidence, with unit incident vector

$$a_{ni} = a_x \sin \theta_i + a_z \cos \theta_i$$

where $\theta_i$ is the angle of incidence measured from the normal to the boundary surface, we have

$$E_i (x, z) = a_y E_{i0} e^{-j\beta_i a_{ni} \cdot R} = a_y E_{i0} e^{-j\beta_1 (x \sin \theta_i + z \cos \theta_i)}$$

$$H_i (x, z) = 1/\zeta_1 [a_{ni} \times E_i (x, z)]$$

$$= E_{i0}/\eta_1 (-a_x \cos \theta_i + a_z \sin \theta_i) e^{-j\beta_1 (x \sin \theta_i + z \cos \theta_i)}$$

For the reflected wave, $a_{nr} = a_x \sin \theta_r - a_z \cos \theta_r$

where $\theta_r$ is the angle of reflection, we have

$$E_r (x, z) = a_y E_{r0} e^{-j\beta_1 (x \sin \theta_r - z \cos \theta_r)}$$
7.7 Oblique incidence at a conducting Plane

Incident Wave:
\[ E_i(x, z) = a_y E_{i0} e^{-j\beta_i a_{ni} \cdot R} = a_y E_{i0} e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)} \]

Reflect Wave:
\[ E_r(x, z) = a_y E_{r0} e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)} \]

The boundary condition
\[ E_i(x, 0) = E_i(x, 0) + E_r(x, 0) = a_y (E_{i0} e^{-j\beta_1 x\sin\theta_i} + E_{r0} e^{-j\beta_1 x\sin\theta_r}) = 0 \]

tells that we must have \( E_{r0} = -E_{i0} \) and matched phase terms, that is,
\[ \theta_i = \theta_r \]

The latter relation is referred to as **Snell's law of reflection**.

Therefore,
\[ E_r(x, z) = -a_y E_{i0} e^{-j\beta_1 (x\sin\theta_i - z\cos\theta_i)} \]
\[ H_r(x, z) = \frac{1}{\eta_1} \left[ a_{nr} \times E_r(x, z) \right] \]
\[ = \frac{\eta_1}{E_{i0}} (-a_x \cos\theta_i - a_z \sin\theta_i) e^{-j\beta_1 (x\sin\theta_i - z\cos\theta_i)} \]
7.7 Oblique incidence at a conducting Plane

The total field is obtained by adding the incident and reflected fields:

\[
\mathbf{E}_i(x, z) = \mathbf{E}_i(x, z) + \mathbf{E}_r(x, z) = a_y j2E_{i0} \sin(\beta_1 z \cos \theta_i)e^{-j\beta_1 x \sin \theta_i}
\]

\[
= -a_y E_{i0} \left( e^{-j\beta_1 z \cos \theta_i} - e^{j\beta_1 z \cos \theta_i} \right)e^{-j\beta_1 x \sin \theta_i}
\]

\[
\mathbf{H}_1(x, z) = -2 \frac{E_{i0}}{\eta_i} \left[ a_x \cos \theta_i \cos(\beta_1 z \cos \theta_i) e^{-j\beta_1 x \sin \theta_i} + a_z j \sin \theta_i \sin(\beta_1 z \cos \theta_i) e^{-j\beta_1 x \sin \theta_i} \right]
\]

Discussions:

1. In z-direction, \(E_{iy}\) and \(H_{Ix}\) maintain standing-wave patterns according to \(\sin(\beta_{1z}z)\) and \(\cos(\beta_{1z}z)\) respectively, where \(\beta_{1z} = \beta_1 \cos \theta_i\). No average power is propagated in this direction since \(E_{iy}\) and \(H_{Ix}\) are 90° out of time phase.

2. In the direction (x-direction) parallel to the boundary, \(E_{iy}\) and \(H_{Ix}\) are in both time and space phase and propagate with a phase velocity

\[
u_{1x} = \frac{\omega}{\beta_{1x}} = \frac{\omega}{\beta_1 \sin \theta_i} = \frac{u_1}{\sin \theta_i}
\]

3. The propagating wave in the x direction is a **nonuniform plane wave** because its amplitude varies with z.
7.7 Oblique incidence at a conducting Plane

**EXAMPLE 8–10** A uniform plane wave \((E_i, H_i)\) of an angular frequency \(\omega\) is incident from air on a very large, perfectly conducting wall at an angle of incidence \(\theta_i\) with perpendicular polarization. Find (a) the current induced on the wall surface, and (b) the time-average Poynting vector in medium 1.

**Solution**

a) The conditions of this problem are exactly those we have just discussed; hence we could use the formulas directly. Let \(z = 0\) be the plane representing the surface of the perfectly conducting wall, and let \(E_i\) be polarized in the \(y\) direction, as was shown in Fig. 8–11. At \(z = 0\), \(E_1(x, 0) = 0\), and \(H_1(x, 0)\) can be obtained

\[
H_1(x, 0) = -\frac{E_i}{\eta_0} (a_x 2 \cos \theta_i) e^{-j\beta_0 x \sin \theta_i}.
\]

Inside the perfectly conducting wall, both \(E_2\) and \(H_2\) must vanish. There is then a discontinuity in the magnetic field. The amount of discontinuity is equal to the surface current. From Eq. (7–68b) we have

\[
J_s(x) = a_{n2} \times H_1(x, 0)
\]

\[
= (-a_z) \times (-a_x) \frac{E_i}{\eta_0} (2 \cos \theta_i) e^{-j\beta_0 x \sin \theta_i}
\]

\[
= a_y \frac{E_i}{60\pi} (\cos \theta_i) e^{-j(\omega/c)x \sin \theta_i}.
\]
7.7 Oblique incidence at a conducting Plane

the surface current. From Eq. (7–68b) we have
\[ J_s(x) = a_n \times H_1(x, 0) \]
\[ = (-a_x) \times (-a_x) \frac{E_{i0}}{\eta_0} (2 \cos \theta_i) e^{-j\beta_0 x \sin \theta_i} \]
\[ = a_y \frac{E_{i0}}{60\pi} (\cos \theta_i) e^{-j(\omega/c)x \sin \theta_i}. \]

The instantaneous expression for the surface current is
\[ J_s(x, t) = a_y \frac{E_{i0}}{60\pi} \cos \theta_i \cos \omega \left( t - \frac{x}{c} \sin \theta_i \right) \text{ (A/m)}. \]

It is this induced current on the wall surface that gives rise to the reflected wave in medium 1 and cancels the incident wave in the conducting wall.

b) The time-average Poynting vector in medium 1 is found by
\[ \mathcal{P}_{av_1} = \frac{1}{2} \Re \left[ E_1(x, z) \times H_1^*(x, z) \right] \]
\[ = a_x 2 \frac{E_{i0}^2}{\eta_1} \sin \theta_i \sin^2 \beta_1 z, \]

where \( \beta_{1z} = \beta_1 \cos \theta_i \). The time-average Poynting vector in medium 2 (a perfect conductor) is, of course, zero.
7.8 Normal Incidence at a Plane Dielectric Boundary

Consider the situation where the incident wave travels in the +z-direction and the boundary surface is the plane \( z = 0 \). The incident \( \mathbf{E} \) and \( \mathbf{H} \) field intensity phasors are

\[
\mathbf{E}_i(z) = a_x E_{i0} e^{-j\beta_1 z} \quad \text{and} \quad \mathbf{H}_i(z) = a_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}
\]

Because of the medium discontinuity at \( z = 0 \), the incident wave is partly reflected back into medium 1 and partly transmitted into medium 2. We have

**a) For the reflected wave (\( \mathbf{E}_r, \mathbf{H}_r \)):**

\[
\mathbf{E}_r(z) = a_x E_{r0} e^{j\beta_1 z} \quad \text{and} \quad \mathbf{H}_r(z) = (-a_z) \times \frac{1}{\eta_1} \mathbf{E}_r(z) = -a_y \frac{E_{i0}}{\eta_1} e^{j\beta_1 z}
\]

**b) For the transmitted wave (\( \mathbf{E}_t, \mathbf{H}_t \)):**

\[
\mathbf{E}_t(z) = a_x E_{t0} e^{-j\beta_2 z} \quad \text{and} \quad \mathbf{H}_t(z) = (a_z) \times \frac{1}{\eta_2} \mathbf{E}_t(z) = a_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z}
\]

where \( E_{t0} \) is the magnitude of \( \mathbf{E}_t \) at \( z = 0 \), and \( \hat{a}_2 \) and \( \zeta_2 \) are the phase constant and the intrinsic impedance, respectively, of medium 2.
7.8 Normal Incidence at a Plane Dielectric Boundary

\[ E_i(z) = a_x E_{i0} e^{-j\beta_1 z} \quad \text{and} \quad H_i(z) = a_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z} \]

\[ E_r(z) = a_x E_{r0} e^{j\beta_1 z} \quad \text{and} \quad H_r(z) = (-a_z) \times \frac{1}{\eta_1} E_r(z) = -a_y \frac{E_{i0}}{\eta_1} e^{j\beta_1 z} \]

\[ E_t(z) = a_x E_{t0} e^{-j\beta_2 z} \quad \text{and} \quad H_t(z) = (a_z) \times \frac{1}{\eta_2} E_t(z) = a_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z} \]

Applying the boundary condition that

\[ E_i(0) + E_r(0) = E_t(0) \quad \text{or} \quad E_{i0}(0) + E_{r0}(0) = E_{t0}(0) \]

\[ H_i(0) + H_r(0) = H_t(0) \quad \text{or} \quad \frac{1}{\eta_1} (E_{i0} - E_{r0}) = E_{t0} / \eta_2 \]

\[ \Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{reflection coefficient} \]

\[ \tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1} \quad \text{transmission coefficient} \]
7.8 Normal Incidence at a Plane Dielectric Boundary

It can be found that \( 1 + \Gamma = \tau \) (Dimensionless)

The total electric field in medium 1 can be written as

\[
E_i(z) = a_x E_{i0} \left( e^{-j\beta_1 z} + \frac{1}{1 + \Gamma} e^{j\beta_1 z} \right) 
\]

\[
H_i(z) = a_y \frac{E_{i0}}{\eta_1} \left( e^{-j\beta_1 z} - \frac{1}{1 - \Gamma} e^{j\beta_1 z} \right) 
\]

Discussions:

1. \( \Gamma > 0 \) (\( \eta_2 > \eta_1 \))

The maximum value of \( |E_i(z)| \) is \( E_{i0}(1 + \Gamma) \)

\[
z_{\text{max}} = -\frac{n\pi}{\beta_1} = -\frac{n\lambda}{2}, \quad n = 1, 2, \ldots
\]

The minimum value of \( |E_i(z)| \) is \( E_{i0}(1 - \Gamma) \), at \( z_{\text{min}} = -\frac{(2n + 1)\pi}{2\beta_1} = -\frac{(2n + 1)\lambda}{4}, \quad n = 1, 2, \ldots
\]

2. \( \Gamma < 0 \) (\( \eta_2 < \eta_1 \))

The location of the maximum value and the minimum value of electric field are interchanged as compared to case 1.

\[
\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{reflection coefficient}
\]

\[
\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1} \quad \text{transmission coefficient}
\]
7.8 Normal Incidence at a Plane Dielectric Boundary

The ratio of the maximum value to the minimum value of the electric field intensity of a standing wave is called the standing-wave ratio (SWR), \( S \).

It can be easily found that

\[
\begin{align*}
\text{when } & \Gamma > 0 (\eta_2 > \eta_1), \quad \eta_2 = S \cdot \eta_1 \\
\text{when } & \Gamma < 0 (\eta_2 < \eta_1), \quad \eta_1 = S \cdot \eta_2 
\end{align*}
\]

\[
S = \frac{|E|_{\text{max}}}{|E|_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad \text{or} \quad |\Gamma| = \frac{S - 1}{S + 1}
\]

While the value of \( \Gamma \) ranges from \(-1\) to \(+1\), the value of \( S \) ranges from \(1\) to \(\infty\). It is customary to express \( S \) on a logarithmic scale. The standing-wave ratio in decibels is \(20 \log_{10} S\). Thus \( S=2 \) corresponds to a standing-wave ratio of \(20 \log_{10} 2 = 6.02 \text{ dB}\) and \(|\Gamma| = (2 - 1)/(2 + 1) = 1/3\). A standing-wave ratio of \(2 \text{ dB}\) is equivalent to \( S = 1.26 \) and \(|\Gamma| = 0.115\).

In medium 2, \((E_t, H_t)\) constitute the transmitted wave propagating in \(+z\)-direction.

\[
\begin{align*}
E_t(z) &= a_x \tau E_i e^{-j\beta_2 z} \\
H_t(z) &= a_y \frac{\tau}{\eta_2} E_i e^{-j\beta_2 z}
\end{align*}
\]
7.8 Normal Incidence at a Plane Dielectric Boundary

We define the **wave impedance of the total field** at any plane parallel to the plane boundary as the ratio of the *total* electric field intensity to the *total* magnetic field intensity. With a $z$-dependent uniform plane wave,

$$Z(z) = \frac{\text{Total } E_x(z)}{\text{Total } H_y(z)} \quad (\Omega)$$

The wave impedance in medium 1 is

$$Z_1(z) = \frac{E_{1x}(z)}{H_{1y}(z)} = \eta_1 e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}$$

At distance $z = -l$ to the left of the boundary plane,

$$Z_1(-l) = \frac{E_{1x}(-l)}{H_{1y}(-l)} = \eta_1 e^{j\beta_1 l} + \Gamma e^{-j\beta_1 l}$$

Using the definition of $\Gamma = (\eta_2 - \eta_1)/(\eta_2 + \eta_1)$ in the equation, we obtain

$$Z_1(-l) = \eta_1 \frac{\eta_2 \cos \beta_1 l + j \eta_1 \sin \beta_1 l}{\eta_1 \cos \beta_1 l + j \eta_2 \sin \beta_1 l}$$

If the plane boundary is perfectly conducting, $\zeta_2 = 0$ and $\Delta = -1$,

$$Z_1(-\ell) = j \eta_1 \tan \beta_1 \ell$$