Reflection of plane waves
Wave impedance of the total field
Reflection of EM waves

- Reflection takes place when an EM wave hits on a boundary. Part of the wave gets reflected, and part of it gets transmitted.

- *Propagation directions* and *amplitudes* of the reflected and transmitted waves depend on *boundary conditions at the interface* and *material properties of the two media* respectively.

- Analysis procedure:
  - Determine expressions of the incident plane wave \((E_i & H_i)\).
  - Determine expressions of the reflected plane wave \((E_r & H_r)\).
  - Determine expressions of the transmitted plane wave \((E_t & H_t)\) if necessary.
    - It depends on intrinsic impedances of the two media.
  - Determine amplitudes of reflected and transmitted plane waves with the aid of reflection and transmission coefficients.
    - Express the amplitudes in terms of incident amplitude, \(E_{i0}\).
  - Determine plane wave expressions in medium 1 and 2.
    - Sum up expressions of incident and reflected waves to give plane wave expression in medium 1.
    - Expression of transmitted plane wave is already the one in medium 2.
Example 1

P.8–22 A uniform sinusoidal plane wave in air with the following phasor expression for electric intensity

\[ E_i(x, z) = a_x 10e^{-j(6x + 8z)} \] (V/m)

is incident on a perfectly conducting plane at \( z = 0 \).

a) Find the frequency and wavelength of the wave.

b) Write the instantaneous expressions for \( E_i(x, z; t) \) and \( H_i(x, z; t) \), using a cosine reference.

c) Determine the angle of incidence.

d) Find \( E_r(x, z) \) and \( H_r(x, z) \) of the reflected wave.

e) Find \( E_t(x, z) \) and \( H_t(x, z) \) of the total field.

[Diagram of incident and reflected waves]
Solution:
\[ \vec{E}_i (x, z) = \vec{\hat{y}} E_{i0} e^{-j \beta_i \vec{n}_i \cdot \vec{R}} = \vec{\hat{y}} E_{i0} e^{-j \beta_i (\frac{3}{5} x + \frac{4}{5} z)} \]
\[ \Rightarrow \vec{n}_i \cdot \hat{\beta}_i \beta_i (\frac{3}{5} x + \frac{4}{5} z) \]
\[ \beta_i \sin \theta_i = \frac{2}{5}, \quad \cos \theta_i = \frac{4}{5} \]
\[ \Rightarrow \sin \theta_i = \frac{2}{5}, \quad \cos \theta_i = \frac{4}{5} \]
\[ (a) \quad \beta_i = 10 = \frac{2 \pi}{\lambda}, \quad \lambda = \frac{2 \pi}{5} \text{ (m)} \]
\[ C = \frac{f \lambda}{\lambda} \Rightarrow f = \frac{C \lambda}{\lambda} = \frac{5 \times 3 \times 10^8}{5} = \frac{15 \times 10^8}{\pi} = 4.78 \times 10^8 \text{ (Hz)} \]
(b) \[ \vec{H}_i (x, z; t) = \vec{\hat{y}} 10 \cos (\omega t - 6x - 8z) = \vec{\hat{y}} 10 \cos (3 \times 10^7 t - 6x - 8z) \]
\[ \vec{H}_i (x, z; t) = \frac{10}{\tilde{\alpha}} (-\tilde{\alpha}_y \cos \theta_i + \tilde{\alpha}_z \sin \theta_i) \cos (\omega t - 6x - 8z) \]
\[ \vec{a}_m \times \vec{a}_y \]
\[ = \left( \frac{3}{5} \tilde{a}_x + \frac{4}{5} \tilde{a}_z \right) \times \vec{a}_y \]
\[ = -\frac{4}{5} \tilde{a}_x + \frac{3}{5} \tilde{a}_z \]
\[ \beta_i \sin \theta_i = 6, \quad \beta_i \cos \theta_i = 8 \Rightarrow \beta_i = \pm 10 \]
\[ \sin^2 \theta_i + \cos^2 \theta_i = 1 \quad \text{but, } \beta_i = \frac{2 \pi}{\lambda_i} > 0 \]
\[ \therefore \beta_i = 10 \]
Example 1 (cont.)

(c) \( \sin \Theta \hat{\imath} = \frac{3}{5} \) \( \Theta \hat{\jmath} = a \hat{\jmath} + \frac{3}{5} \hat{\imath} = 36.9^\circ \)

(d) \( \vec{E}_r(x,z) = \hat{\imath} E_0 e^{-j \beta \left( x \sin \theta - z \cos \theta \right)} = -a \hat{\jmath} e^{j \beta \left( x \sin \theta - z \cos \theta \right)} \)
\[\Theta \hat{r} = \Theta \hat{\jmath} \]
\[\vec{H}_r(x,z) = - (\hat{\jmath} \frac{a}{15 \pi} + \hat{\imath} \frac{a}{20 \pi}) e^{-j(6x-8z)} \]

(e) \( E_1(x,z) = E_i(x,z) + E_r(x,z) \)
\[= a_y 10 e^{-j 6x} (e^{-j 8x} - e^{j 8x}) = a_y 10 e^{-j 6x} (-j 2 \sin 8z) \]
\[= a_y 20 (\sin 8z) e^{-j \left( \frac{6x + \pi}{2} \right)} \]

\( H_1(x,z) = H_i(x,z) + H_r(x,z) \)
\[= -a_x \frac{1}{15 \pi} e^{-j 6x} (e^{-j 8x} + e^{j 8x}) + a_y \frac{1}{20 \pi} e^{-j 6x} (e^{-j 8x} - e^{j 8x}) \]
\[= -a_x \frac{1}{15 \pi} (2 \cos 8z) e^{-j 6x} + a_y \frac{1}{20 \pi} (-2j \sin 8z) e^{-j 6x} \]
\[= -a_x \frac{2}{15 \pi} (\cos 8z) e^{-j 6x} + a_y \frac{1}{10 \pi} (\sin 8z) e^{-j \left( \frac{6x + \pi}{2} \right)} \]
Example 2

P.8–24 For the case of oblique incidence of a uniform plane wave with perpendicular polarization on a perfectly conducting plane boundary as shown in Fig. 8–11, write (a) the instantaneous expressions

\[ \mathbf{E}_1(x, z; t) \text{ and } \mathbf{H}_1(x, z; t) \]

for the total field in medium 1, using a cosine reference, and (b) the time-average Poynting vector.

FIGURE 8–11
Plane wave incident obliquely on a plane conducting boundary (perpendicular polarization).
Example 2 (cont.)

Solution:

\[
E_i = a_y E_{i0} e^{-j\beta_i (x \sin \theta_i + z \cos \theta_i)}
\]

\[
H_i = \frac{E_{i0}}{\eta_1} (-a_x \cos \theta_i + a_z \sin \theta_i) e^{-j\beta_i (x \sin \theta_i + z \cos \theta_i)}
\]

\[
E_r = -a_y E_{i0} e^{-j\beta_i (x \sin \theta_i - z \cos \theta_i)}
\]

\[
H_r = -\frac{E_{i0}}{\eta_1} (a_x \cos \theta_i + a_z \sin \theta_i) e^{-j\beta_i (x \sin \theta_i - z \cos \theta_i)}
\]

\[
E_1 = E_i + E_r = a_y E_{i0} e^{-j\beta_i x \sin \theta_i} \left( e^{-j\beta_i z \cos \theta_i} - e^{j\beta_i z \cos \theta_i} \right) = -a_y 2 j E_{i0} \left[ \sin(\beta_1 z \cos \theta_i) \right] e^{-j\beta_i x \sin \theta_i} \quad \therefore \quad \theta_i = \theta_r
\]

\[
H_1 = H_i + H_r = \frac{E_{i0}}{\eta_1} \left[ -a_x \cos \theta_i e^{-j\beta_i x \sin \theta_i} \left( e^{-j\beta_i z \cos \theta_i} + e^{j\beta_i z \cos \theta_i} \right) + a_z \sin \theta_i e^{-j\beta_i x \sin \theta_i} \left( e^{-j\beta_i z \cos \theta_i} - e^{j\beta_i z \cos \theta_i} \right) \right]
\]

\[
= \frac{E_{i0}}{\eta_1} e^{-j\beta_i x \sin \theta_i} \left\{ -a_x \cos \theta_i [2 \cos(\beta_1 z \cos \theta_i)] + a_z \sin \theta_i [2 j \sin(\beta_1 z \cos \theta_i)] \right\}
\]

(a) \[
E_1 = a_y 2 E_{i0} \left[ \sin(\beta_1 z \cos \theta_i) \right] \sin(\omega t - \beta_i x \sin \theta_i)
\]

\[\text{(instantaneous form)}\]

\[
H_1 = \frac{2 E_{i0}}{\eta_1} \left\{ -a_x \cos \theta_i \left[ \cos(\beta_1 z \cos \theta_i) \right] \cos(\omega t - \beta_i x \sin \theta_i) + a_z \sin \theta_i \left[ \sin(\beta_1 z \cos \theta_i) \right] \sin(\omega t - \beta_i x \sin \theta_i) \right\}
\]

(b) \[
E_1 \times H_1^* = a_x E_y H_z^* - a_z E_y H_x^*
\]

\[\because \quad P_{av1} = \frac{1}{2} \text{Re}(E_1 \times H_1^*) = \frac{2 E_{i0}^2}{\eta_1} a_x \sin \theta_i \sin^2(\beta_1 z \cos \theta_i) \quad P_{av2} = \frac{1}{2} \text{Re}(E_2 \times H_2^*) = 0\]
Reflection of EM waves

- Consider normal incidence at a plane dielectric boundary

**Reflection coefficient**

\[
\Gamma = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}
\]

**Transmission coefficient**

\[
\tau = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_2 + \eta_1}
\]

i.e.: \( 1 + \Gamma = \tau \)

- Properties:
  - \( \Gamma \) is dimensionless.
  - It can either be positive, negative or complex. A complex \( \Gamma \) (or \( \tau \)) means a *phase shift* is introduced at the interface upon reflection (or transmission).

- Consider plane wave expression in medium 1:

\[
\mathbf{E}_1(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z) = a_x E_i \left( e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z} \right)
\]

\[
= a_x E_i \left[ (1 + \Gamma)e^{-j\beta_1 z} + \Gamma \left( e^{j\beta_1 z} - e^{-j\beta_1 z} \right) \right]
\]

\[
= a_x E_i \left[ \tau e^{-j\beta_1 z} + \Gamma \left( j2\sin\beta_1 z \right) \right]
\]

\[
= a_x E_i \left[ \tau \cos(\omega t - \beta_1 z) + 2\Gamma \sin\beta_1 z \sin\omega t \right]
\]

(standing wave

\[
\text{Travelling wave} \quad \text{Standing wave (energy does not propagate)}
\]
Example 3

Suppose a plane wave polarized in y-direction and propagating in x-direction is incident \textit{normally} from medium 1 (\(\varepsilon_1, \mu_1\)) to medium 2 (\(\varepsilon_2, \mu_2\)) (both are lossless) at \(x = 0\). Amplitude of the electric field intensity in medium 1 is \(E_i\) while frequency and initial phase is \(f_0\) and zero respectively. Find:

(a) intrinsic impedances \(\eta\) and phase constant \(\beta\) of the two media.

(b) instantaneous form of electric and magnetic field intensities in the two media. Express them in terms on \(\eta, \beta, \Gamma, \tau\).
Example 3 (cont.)

Solution:

(a) Intrinsic impedance of medium 1 and 2 are respectively

\[ \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} \quad \text{and} \quad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \]

Phase constant of a medium characterized by \( \varepsilon_1 \) and \( \mu \) is

\[ \beta = k = \omega \sqrt{\mu \varepsilon} = 2\pi f_0 \sqrt{\mu \varepsilon} \]

Therefore, phase constant of medium 1 and 2 are respectively

\[ \beta_1 = 2\pi f_0 \sqrt{\mu_1 \varepsilon_1} \quad \text{and} \quad \beta_2 = 2\pi f_0 \sqrt{\mu_2 \varepsilon_2} \]

(b) \[ E_1 = E_i + E_r \]

\[ = a_y E_i e^{-j\beta_1 x} + a_y E_r e^{j\beta_1 x} \]

\[ = a_y E_i \left( e^{-j\beta_1 x} + \Gamma e^{j\beta_1 x} \right) \]

\[ = a_y E_i \left[ \tau \cos(\omega t - \beta_1 x) + 2\Gamma \sin \beta_1 x \sin \omega t \right] \quad \text{(from slide 8)} \]

\[ E_2 = E_i \]

\[ = a_y E_i \tau e^{-j\beta_2 x} \]

\[ = a_y E_i \tau \cos(\omega t - \beta_2 x) \]
Example 3 (cont.)

(b) (cont.)

\[ H_1 = H_i + H_r \]
\[ = \frac{1}{\eta_1} (a_x \times a_y) E_i e^{-j\beta_1 x} + \frac{1}{\eta_1} (-a_x \times a_y) E_r e^{j\beta_1 x} \]
\[ = a_z \frac{E_i}{\eta_1} (e^{-j\beta_1 x} - \Gamma e^{j\beta_1 x}) \]
\[ = a_z \frac{E_i}{\eta_1} [(1 - \Gamma)e^{-j\beta_1 x} - \Gamma(e^{j\beta_1 x} - e^{-j\beta_1 x})] \]
\[ = a_z \frac{E_i}{\eta_1} [(1 - \Gamma)e^{-j\beta_1 x} - \Gamma(j2 \sin \beta_1 x)] \]
\[ = a_z \frac{E_i}{\eta_1} [(1 - \Gamma)\cos(\omega t - \beta_1 x) - 2\Gamma \sin \beta_1 x \sin \omega t] \]

\[ H_2 = H_t \]
\[ = \frac{1}{\eta_2} (a_x \times a_y) E_t e^{-j\beta_2 x} \]
\[ = a_y \frac{E_t \tau}{\eta_2} e^{-j\beta_2 x} \]

Steps are similar to those in slide 8.
Wave impedance of the total field

Wave impedance at a plane parallel to the boundary plane along z-axis is defined by:

\[
Z(z) = \frac{|E(z)|}{|H(z)|}
\]

Suppose, in medium 1,

\[
|E_1| = E_i e^{-j\beta_1 z} + E_r e^{j\beta_1 z}
\]

\[
|H_1| = \frac{E_i}{\eta_1} \left( e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z} \right)
\]

Therefore, wave impedance in medium 1,

\[
Z_1(z) = \eta_1 \frac{e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}}{e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z}}
\]

Because we only concern \(Z\) in medium 1, it’s more convenient to parameterize \(Z\) by a positive variable. We put \(z = -l\). Then,

\[
Z_1(-l) = \eta_1 \frac{e^{j\beta_1 l} + \Gamma e^{-j\beta_1 l}}{e^{j\beta_1 l} - \Gamma e^{-j\beta_1 l}} \text{, where } \Gamma = \Gamma_1(z = 0) = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}
\]

\[
Z_1(-l) = \eta_1 \frac{\eta_2 \cos \beta_1 l + j \eta_1 \sin \beta_1 l}{\eta_1 \cos \beta_1 l + j \eta_2 \sin \beta_1 l}
\]
Example 4

A plane wave of $\lambda = 3$ cm, in free-space, is incident normally on a sheet of substrate ($\varepsilon_r = 4.9$, $\alpha = 0$), find:

(a) thickness of the sheet such that no reflection occurs.

(b) ratio of transmitted to incident power if wave frequency is decreased by 10%.

Let medium 1 and 3 be free-space and medium 2 be the substrate.
Example 4 (cont.)

(a) Solution: In medium 1, \[ Z_1(-l) = \frac{\eta_2 \cos \beta_1 l + j \eta_1 \sin \beta_1 l}{\eta_1 \cos \beta_1 l + j \eta_2 \sin \beta_1 l} \]

Similarly, in medium 2, \[ Z_2(z = 0) = \frac{\eta_3 \cos \beta_2 d + j \eta_2 \sin \beta_2 d}{\eta_2 \cos \beta_2 d + j \eta_3 \sin \beta_2 d} \]

For no reflection, we put \[ \Gamma_1(z = 0) = \frac{Z_2(z = 0) - \eta_1}{Z_2(z = 0) + \eta_1} = 0 \]

Therefore, \[ \eta_2 (\eta_3 \cos \beta_2 d + j \eta_2 \sin \beta_2 d) = \eta_1 (\eta_2 \cos \beta_2 d + j \eta_3 \sin \beta_2 d) \]

By comparing real and imaginary part of both sides, we have:

\[
\begin{cases}
\eta_2 \eta_3 \cos \beta_2 d = \eta_1 \eta_2 \cos \beta_2 d \implies (\eta_1 - \eta_3) \cos \beta_2 d = 0 \quad \text{(rejected since } \eta_1 = \eta_3) \\
\eta_1 \eta_2 \sin \beta_2 d = \eta_1 \eta_3 \sin \beta_2 d \implies (\eta_2^2 - \eta_1 \eta_3) \sin \beta_2 d = 0
\end{cases}
\]

Suppose \( \eta_2^2 \neq \eta_1 \eta_3 \),

\[ \sin \beta_2 d = 0 \implies \beta_2 d = n \pi \implies d = \frac{n \lambda_2}{2} \]

Put \( n = 1 \), \[ d = \frac{\lambda_2}{2} = \frac{13.533}{2} = 6.777 \text{ mm} \]

\[ f = \frac{c}{\lambda_1} = \frac{3 \times 10^8}{3 \times 10^{-2}} = 10^{10} \text{ Hz} \]

\[ u_2 = \frac{1}{\sqrt{\mu_2 \varepsilon_2}} = \frac{c}{\sqrt{\varepsilon_{r2}}} = \frac{3 \times 10^8}{\sqrt{4.9}} = 135.5 \times 10^6 \text{ m/s} \]

\[ \lambda_2 = \frac{u_2}{f} = \frac{135.5 \times 10^6}{10^{10}} = 13.553 \text{ mm} \]
Example 4 (cont.)

\( \eta_1 = \eta_3 = 120\pi \Omega \)
\( \eta_2 = \sqrt{\frac{\mu \mu_0}{\varepsilon \varepsilon_0}} = \frac{120\pi}{\sqrt{4.9}} = 170.30723 \Omega \)
\( f = 0.9(10^{10}) = 9 \times 10^9 \) Hz

For E- and H-field in medium 2, we have:

\[
\begin{align*}
E_2^+ e^{-j\beta_2 d} + E_2^- e^{j\beta_2 d} &= E_3^+ e^{-j\beta_3 d} \\
\frac{1}{\eta_2} \left( E_2^+ e^{-j\beta_2 d} - E_2^- e^{j\beta_2 d} \right) &= \frac{E_3^+}{\eta_3} e^{-j\beta_3 d}
\end{align*}
\]

\[
\eta_2 \left( 1 + \Gamma_2 e^{j2\beta_2 d} \right) = \eta_3 , \text{ where } \Gamma_2 = \Gamma_2(z = 0) = \frac{E_2^-}{E_2^+}
\]

\[
\Gamma_2 = \frac{\eta_2 - \eta_3}{\eta_2 + \eta_3} e^{-j2\beta_2 d} = 0.38 e^{-j5.67}
\]

At \( z = 0 \),

\[
\begin{align*}
E_1^+ + E_1^- &= E_2^+ + E_2^- \\
\frac{1}{\eta_1} (E_1^+ - E_1^-) &= \frac{1}{\eta_2} (E_2^+ - E_2^-)
\end{align*}
\]

\[
\eta_1 \frac{E_1^+ + E_1^-}{E_1^+ - E_1^-} = \eta_2 \frac{1 + \Gamma_2}{1 - \Gamma_2}
\]

\[
E_1^+ \left( \frac{1 + \Gamma_2}{1 - \Gamma_2} - \frac{\eta_1}{\eta_2} \right) = E_1^- \left( \frac{1 + \Gamma_2}{1 - \Gamma_2} + \frac{\eta_1}{\eta_2} \right)
\]

Then,

\[
\frac{E_1^-}{E_1^+} = \frac{1 - \sqrt{4.9} + \Gamma_2 \left( 1 + \sqrt{4.9} \right)}{1 + \sqrt{4.9} + \Gamma_2 \left( 1 - \sqrt{4.9} \right)} = 0.26 e^{j1.95} \Rightarrow \left| \frac{E_1^-}{E_1^+} \right| = 0.26
Example 4 (cont.)

(b) (cont.)

Let $S^+_{av}$ and $S^-_{av}$ be average incident and reflected power respectively, such that:

$$S^+_{av} = \frac{|E^+_1|^2}{2\eta_1} \quad \text{and} \quad S^-_{av} = \frac{|E^-_1|^2}{2\eta_1}$$

Let $S^+_{av2}$ be average transmitted power, such that:

$$S^+_{av2} = S^+_{av1} - S^-_{av1} = \frac{|E^+_1|^2 - |E^-_1|^2}{2\eta_1}$$

Therefore,

$$\frac{S^+_{av2}}{S^+_{av1}} = \frac{|E^+_1|^2 - |E^-_1|^2}{|E^+_1|^2} = 1 - \frac{|E^-_1|^2}{|E^+_1|^2} = 1 - \frac{|E^-_1|^2}{|E^+_1|^2} = 1 - 0.26^2 = 0.93$$