3.5 Method of Images

Method of images replaces the original boundary by appropriate image charges in lieu of a formal solution of Poisson’s or Laplace’s equation so that the original problem is greatly simplified.

The basic principle of the method of images is the uniqueness theorem. As long as (1) the solution satisfies Poisson’s or Laplace’s equation and (2) the solution satisfies the given boundary condition, the simplest solution should be taken.

**Point charge over grounded plane conductor**

By method of images

By direct solution

V(x, y, z) = \( \frac{Q}{4\pi\varepsilon_0 \sqrt{x^2 + (y-d)^2 + z^2}} + \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_s}{R_1} ds \)

where \( R_1 \) is the distance from \( ds \) to the point under consideration and \( S \) is the surface of the entire conducting plane.

Only valid in the region of \( y > 0 \).
**Example** A positive point charge $Q$ is located at distances $d_1$ and $d_2$, respectively, from two grounded perpendicular conducting half-planes, as shown in the figure. Determine the force on $Q$ caused by the charges induced on the planes.

\[
\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3, \\
\mathbf{F}_1 = -a_y \frac{Q^2}{4\pi \varepsilon_0 (2d_2)^2}, \\
\mathbf{F}_2 = -a_x \frac{Q^2}{4\pi \varepsilon_0 (2d_1)^2}, \\
\mathbf{F}_3 = \frac{Q^2}{4\pi \varepsilon_0 [(2d_1)^2 + (2d_2)^2]^{3/2}} (a_x 2d_1 + a_y 2d_2).
\]
3.5 Method of Images

Line Charge And Parallel Conducting Cylinder

Let’s take a trial solution (an intelligent guess) that \( \rho_i = -\rho_l \)

We have found that the \( \mathbf{E} \) field generated by a line charge is

\[
\mathbf{E} = a_r \frac{\rho_l}{2\pi\varepsilon_0 r} \quad (V/m)
\]

The electric potential at a distance \( r \) from a line charge of density \( \rho_l \) can be obtained by integrating the electric field intensity \( \mathbf{E} \)

\[
V = -\int_{r_0}^{r} E_r \, dr = -\frac{\rho_l}{2\pi\varepsilon_0} \int_{r_0}^{r} \frac{1}{r} \, dr = \frac{\rho_l}{2\pi\varepsilon_0} \ln \frac{r_0}{r}
\]

Note that the reference point for zero potential, \( r_0 \), cannot be at infinity. Let us leave \( r_0 \) unspecified for the time being.
3.5 Method of Images

The potential at a point on or outside the cylindrical surface is obtained by adding the contributions of $\rho_l$ and $\rho_i$. In particular, at a point M on the cylindrical surface, we have

$$V_M = \frac{\rho_l}{2\pi\varepsilon_0} \ln \frac{r_0}{r} - \frac{\rho_l}{2\pi\varepsilon_0} \ln \frac{r_0}{r_i} = \frac{\rho_l}{2\pi\varepsilon_0} \ln \frac{r_i}{r}$$

We have simplified the solution by considering the $r_0$ is so large that the distance of the reference point to $\rho_l$ and $\rho_i$ is negligible.

Equipotential surfaces are specified by $\frac{r_i}{r} = \text{Constant}$

Therefore, triangles $OMP_i$ and $OPM$ similar. We have $\frac{P_iM}{PM} = \frac{OP_i}{OM} = \frac{OM}{OP}$

or $\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant}$

From the above equation we see that if $d_i = \frac{a^2}{d}$ the image line charge $-\rho_i$, together with $\rho_l$, will make the dashed cylindrical surface in the figure equipotential. As the point M changes its location on the dashed circle, both $r_i$ and $r$ will change; but their ratio remains a constant that equals $a/d$. 
3.5 Method of Images

Two Parallel Conducting Cylinders of The Same Radius

Since
\[ V_M = \frac{\rho_i}{2\pi \varepsilon_0} \ln \frac{r_i}{r} \]
and
\[ \frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} \]

We have
\[ V_2 = \frac{\rho_i}{2\pi \varepsilon_0} \ln \frac{a}{d} \quad \text{and} \quad V_1 = -\frac{\rho_i}{2\pi \varepsilon_0} \ln \frac{a}{d} \]

The capacitance per unit length is
\[ C = \frac{\rho_i}{V_1 - V_2} = \frac{\pi \varepsilon_0}{\ln(d/a)} \]

where \( d = D - d_i = D - \frac{a^2}{d} \) from which we obtain \( d = \frac{1}{2}(D + \sqrt{D^2 - 4a^2}) \)

Consequently,
\[ C = \frac{\pi \varepsilon_0}{\ln[(D/2a) + \sqrt{(D/2a)^2 - 1}]} = \frac{\pi \varepsilon_0}{\cosh^{-1}(D/2a)} \quad (F/m) \]
3.5 Method of Images

Two Parallel Conducting Cylinders of The Same Radius (cont.)

\[ V_P = \frac{\rho t}{2\pi\varepsilon_0} \ln \frac{r_2}{r_1}. \]

In the \(xy\)-plane the equipotential lines are defined by \(r_2/r_1 = k\) (constant). We have

\[ \frac{r_2}{r_1} = \frac{\sqrt{(x+b)^2 + y^2}}{\sqrt{(x-b)^2 + y^2}} = k, \]  

which reduces to

\[ \left( x - \frac{k^2 + 1}{k^2 - 1} b \right)^2 + y^2 = \left( \frac{2k}{k^2 - 1} b \right)^2 \]

Equation (4–49) represents a family of circles in the \(xy\)-plane with radii

\[ a = \left| \frac{2kb}{k^2 - 1} \right|. \]
3.6 Method of Separation of Variables

We now develop a method for solving 3-D problems where the boundaries, over which the potential or its normal derivative is specified, coincide with the coordinate surfaces of an orthogonal, curvilinear coordinate system. In such cases, the solution can be expressed as a product of three one-dimensional functions, each depending separately on one coordinate variable only, The procedure is called the **method of separation of variables**.

Classifications of boundary-value problems:

**Dirichlet problems**, in which the value of the potential is specified everywhere on the boundaries;

**Neumann problems**, in which the normal derivative of the potential is specified everywhere on the boundaries;

**Mixed boundary-value problems**, *in which the potential is specified over some boundaries and the normal derivative of the potential is specified over the remaining ones.*
3.6 Method of Separation of Variables

Let’s investigate Laplace’s equation for scalar electric potential $V$ in Cartesian coordinates system first:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

To apply the method of separation of variables, we assume $V(x, y, z) = X(x)Y(y)Z(z)$

Therefore,

$$Y(y)Z(z)\frac{d^2 X(x)}{dx^2} + X(x)Z(z)\frac{d^2 Y(y)}{dy^2} + X(x)Y(y)\frac{d^2 Z(z)}{dz^2} = 0$$

That is

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2} = 0$$

I order for the above equation to be satisfied for all values of $x, y, z$, we must have:

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = -k_x^2, \quad \frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} = -k_y^2, \quad \frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2} = -k_z^2$$

Consequently,

$$\frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0, \quad \frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0, \quad \frac{d^2 Z(z)}{dz^2} + k_z^2 Z(z) = 0$$

and $k_x^2 + k_y^2 + k_z^2 = 0$. 

Basic Electromagnetics, Dept. of Elec. Eng., The Chinese University of Hong Kong, Prof. K.-L. Wu / Prof. Th. Blu Lesson 11–13
3.6 Method of Separation of Variables

Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

<table>
<thead>
<tr>
<th>$k_x^2$</th>
<th>$k_x$</th>
<th>$X(x)$</th>
<th>Exponential forms of $X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$A_0 x + B_0$</td>
<td>$C_1 e^{j k x} + D_1 e^{-j k x}$</td>
</tr>
<tr>
<td>+ $k$</td>
<td></td>
<td>$A_1 \sin k x + B_1 \cos k x$</td>
<td></td>
</tr>
<tr>
<td>- $j k$</td>
<td></td>
<td>$A_2 \sinh k x + B_2 \cosh k x$</td>
<td>$C_2 e^{k x} + D_2 e^{-k x}$</td>
</tr>
</tbody>
</table>

† The exponential forms of $X(x)$ are related to the trigonometric and hyperbolic forms listed in the third column by the following formulas:

$$e^{\pm j k x} = \cos k x \pm j \sin k x, \quad \cos k x = \frac{1}{2} (e^{j k x} + e^{-j k x}), \quad \sin k x = \frac{1}{2j} (e^{j k x} - e^{-j k x});$$

$$e^{\pm j k x} = \cosh k x \pm \sinh k x, \quad \cosh k x = \frac{1}{2} (e^{k x} + e^{-k x}), \quad \sinh k x = \frac{1}{2} (e^{k x} - e^{-k x}).$$

Constant $k$ and the function forms are determined by the given boundary conditions. For example, if the potential $V$ approaches to 0 when $x$ approaches to infinite, the possible solution form is $D e^{-k x}$ with a positive real $k$. 
3.6 Method of Separation of Variables

EXAMPLE Two grounded, semi-infinite, parallel-plane electrodes are separated by a distance $b$. A third electrode perpendicular to and insulated from both is maintained at a constant potential $V_0$. Determine the potential distribution in the region enclosed by the electrodes.

Solution:

With $V$ independent of $z$, we have

$$V(x, y, z) = V(x, y), \quad Z(z) = B_0$$

In the $x$-direction:

$$V(0, y) = V_0, \quad V(\infty, y) = 0$$

In the $y$-direction:

$$V(x, 0) = 0, \quad V(x, b) = 0$$

Since $V(\infty, y) = 0$, we choose $X(x) = D_2 e^{-kx}$, where $k$ is a positive real number and

$$k_y^2 = -k_x^2 = k^2 \implies k_x = jk, \quad k_y = k$$

The boundary conditions in $y$-direction suggest that $Y(y) = A_1 \sin(ky)$

An appropriate solution of the Laplace’s equation satisfying partial boundary conditions (namely, harmonic functions) is

$$V_n(x, y) = B_0 D_2 A_1 e^{-kx} \frac{\sin(ky)}{C_n}$$
3.6 Method of Separation of Variables

\[ V_n(x, y) = B_0 D_2 A_1 e^{-kx} \sin(ky) \]

The constant \( k \) is determined by the BC at \( y=b \), that is

\[ V_n(x, b) = C_n e^{-kx} \sin(kb) = 0 \quad \Rightarrow \quad \sin(kb) = 0 \quad \Rightarrow \quad k = n\pi / b, \quad n = 1, 2, 3, \ldots \]

Therefore, the harmonic function becomes

\[ V_n(x, y) = C_n e^{-n\pi x/b} \sin\left(\frac{n\pi y}{b}\right) \]

In order to satisfy the BC at \( x=0 \), we use the principle of linear superposition of \( V_n \) to find the specific solution of the given BC:

\[ V(x, y) = \sum_{n=1}^{\infty} V_n(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/b} \sin\left(\frac{n\pi y}{b}\right) \quad \Rightarrow \quad V_n(0, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{b}\right) = V_0, \quad 0 < y < b \]

In order to evaluate the coefficients \( C_n \), we multiply both sides of the equation by \( \sin\left(\frac{m\pi y}{b}\right) \) and integrate the products from \( y=0 \) to \( y=b \):

\[ \sum_{n=1}^{\infty} \int_{0}^{b} C_n \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi y}{b}\right) dy = V_0 \int_{0}^{b} \sin\left(\frac{m\pi y}{b}\right) dy \]
3.6 Method of Separation of Variables

\[
\sum_{n=1}^{\infty} \int_{0}^{b} C_n \sin\left(\frac{n\pi}{b} y\right) \sin\left(\frac{m\pi}{b} y\right) \, dy = V_0 \int_{0}^{b} \sin\left(\frac{m\pi}{b} y\right) \, dy
\]

\[
\int_{0}^{b} C_n \sin\left(\frac{n\pi}{b} y\right) \sin\left(\frac{m\pi}{b} y\right) \, dy = \frac{C_n}{2} \int_{0}^{b} \left[ \cos\left(\frac{(n-m)\pi}{b} y\right) - \cos\left(\frac{(n+m)\pi}{b} y\right) \right] \, dy
\]

\[
\begin{cases}
C_n b & \text{if } m = n, \\
0 & \text{if } m \neq n.
\end{cases}
\]

\[
\int_{0}^{b} V_0 \sin\left(\frac{m\pi}{b} y\right) \, dy = \begin{cases}
\frac{2bV_0}{m\pi} & \text{if } m \text{ is odd}, \\
0 & \text{if } m \text{ is even}.
\end{cases}
\]

\[
V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/b} \sin\left(\frac{n\pi}{b} y\right)
\]

\[
= \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n} e^{-n\pi x/b} \sin\left(\frac{n\pi}{b} y\right),
\]

\[
n = 1, 3, 5, \ldots,
\]

\[
x > 0 \quad \text{and} \quad 0 < y < b.
\]
3.6 Method of Separation of Variables

Method of separation of variables in Cylindrical coordinates

For problems with circular cylindrical boundaries we write the governing equations in the cylindrical coordinate system. Laplace’s equation for scalar electric potential $V$ in cylindrical coordinates is, from Eq. (4–8),

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0.
$$

(4–115)

A general solution of Eq. (4–115) requires the knowlege of Bessel functions, the discussion of which will be deferred until Chapter 10. In situations in which the lengthwise dimension of the cylindrical geometry is large in comparison to its radius, the associated field quantities may be considered to be approximately independent of $z$. In such cases, $\frac{\partial^2 V}{\partial z^2} = 0$ and Eq. (4–115) becomes the governing equation of a two-dimensional problem:

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} = 0.
$$

(4–116)

Applying the method of separation of variables, we assume a product solution

$$
V(r, \phi) = R(r)\Phi(\phi),
$$

(4–117)

where $R(r)$ and $\Phi(\phi)$ are, respectively, functions of $r$ and $\phi$ only. Substituting solution (4–117) in Eq. (4–116) and dividing by $R(r)\Phi(\phi)$, we have

$$
\frac{r}{R(r)} \frac{d}{dr} \left[ r \frac{dR(r)}{dr} \right] + \frac{1}{\Phi(\phi)} \frac{d^2 \Phi(\phi)}{d\phi^2} = 0.
$$

(4–118)
3.6 Method of Separation of Variables

In Eq. (4–118) the first term on the left side is a function of $r$ only, and the second term is a function of $\phi$ only. (Note that ordinary derivatives have replaced partial derivatives.) For Eq. (4–118) to hold for all values of $r$ and $\phi$, each term must be a constant and be the negative of the other. We have

$$\frac{r}{R(r)} \frac{d}{dr} \left[ r \frac{dR(r)}{dr} \right] = k^2$$

(4–119)

and

$$\frac{1}{\Phi(\phi)} \frac{d^2\Phi(\phi)}{d\phi^2} = -k^2,$$

(4–120)

where $k$ is a separation constant.

Equation (4–120) can be rewritten as

$$\frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0.$$

(4–121)

This is of the same form as Eq. (4–86), and its solution can be any one of those listed in Table 4–1. For circular cylindrical configurations, potential functions and therefore $\Phi(\phi)$ are periodic in $\phi$, and the hyperbolic functions do not apply. In fact, if the range of $\phi$ is unrestricted, $k$ must be an integer. Let $k$ equal $n$. The appropriate solution is

$$\Phi(\phi) = A_{\phi} \sin n\phi + B_{\phi} \cos n\phi,$$

(4–122)

where $A_{\phi}$ and $B_{\phi}$ are arbitrary constants.
3.6 Method of Separation of Variables

We now turn our attention to Eq. (4–119), which can be rearranged as

$$r^2 \frac{d^2 R(r)}{dr^2} + r \frac{dR(r)}{dr} - n^2 R(r) = 0,$$  \hspace{1cm} (4–123)

where integer $n$ has been written for $k$, implying a $2\pi$ range for $\phi$. The solution of Eq. (4–123) is

$$R(r) = A_r r^n + B_r r^{-n}.$$  \hspace{1cm} (4–124)

This can be verified by direct substitution. Taking the product of the solutions in (4–122) and (4–124), we obtain a general solution of $z$-independent Laplace’s equation (4–116) for circular cylindrical regions with an unrestricted range for $\phi$:

$$V_n(r, \phi) = r^n(A_n \sin n\phi + B_n \cos n\phi) + r^{-n}(A'_n \sin n\phi + B'_n \cos n\phi), \quad n \neq 0.$$  \hspace{1cm} (4–125)

Depending on the boundary conditions the complete solution of a problem may be a summation of the terms in Eq. (4–125). It is useful to note that, when the region of interest includes the cylindrical axis where $r = 0$, the terms containing the $r^{-n}$ factor cannot exist. On the other hand, if the region of interest includes the point at infinity, the terms containing the $r^n$ factor cannot exist, since the potential must be zero as $r \to \infty$. 

3.6 Method of Separation of Variables

Eq. (4–121) has the simplest form when \( k = 0 \). We have

\[
\frac{d^2 \Phi(\phi)}{d\phi^2} = 0.
\]  

(4–126)

The general solution of Eq. (4–126) is \( \Phi(\phi) = A_0 \phi + B_0 \). If there is no circumferential variation, \( A_0 \) vanishes, and we have

\[
\Phi(\phi) = B_0, \quad k = 0.
\]  

(4–127)

The equation for \( R(r) \) also becomes simpler when \( k = 0 \). We obtain from Eq. (4–119)

\[
\frac{d}{dr} \left[ r \frac{dR(r)}{dr} \right] = 0,
\]  

(4–128)

which has a solution

\[
R(r) = C_0 \ln r + D_0, \quad k = 0.
\]  

(4–129)

The product of Eqs. (4–127) and (4–129) gives a solution that is independent of either \( z \) or \( \phi \):

\[
V(r) = C_1 \ln r + C_2,
\]  

(4–130)

where the arbitrary constants \( C_1 \) and \( C_2 \) are determined from boundary conditions.
3.6 Method of Separation of Variables

**Example 4–8** Consider a very long coaxial cable. The inner conductor has a radius \(a\) and is maintained at a potential \(V_0\). The outer conductor has an inner radius \(b\) and is grounded. Determine the potential distribution in the space between the conductors.

**Solution** Figure 4–18 shows a cross section of the coaxial cable. We assume no \(z\)-dependence and, by symmetry, also no \(\phi\)-dependence \((k = 0)\). Therefore, the electric potential is a function of \(r\) only and is given by Eq. (4–130).

The boundary conditions are

\[
V(b) = 0, \quad (4–131a)
\]
\[
V(a) = V_0. \quad (4–131b)
\]

Substitution of Eqs. (4–131a) and (4–131b) in Eq. (4–130) leads to two relations:

\[
C_1 \ln b + C_2 = 0, \quad (4–132a)
\]
\[
C_1 \ln a + C_2 = V_0. \quad (4–132b)
\]

From Eqs. (4–132a) and (4–132b), \(C_1\) and \(C_2\) are readily determined:

\[
C_1 = -\frac{V_0}{\ln (b/a)}, \quad C_2 = \frac{V_0 \ln b}{\ln (b/a)}.
\]

Therefore, the potential distribution in the space \(a \leq r \leq b\) is

\[
V(r) = \frac{V_0}{\ln (b/a)} \ln \left(\frac{b}{r}\right). \quad (4–133)
\]

Obviously, equipotential surfaces are coaxial cylindrical surfaces.